Math 4317
Final Exam
Summer 1993

Instructions:

1. Please do any 6 of the following 7 problems. Indicate clearly on the first page of your exam which 6 problems you would like graded.
2. Explain your work carefully.
3. You may use your class notes and the text by Bartle, except for the "hints" section at the end.
4. Please begin each problem on a new sheet of paper.

1. Let f and g be bounded real valued functions with domain D. Prove that

$$\sup\{f(x) + g(x) : x \quad D\} \leq \sup\{f(x) : x \quad D\} + \sup\{f(x) : x \quad D\}.$$

   Give an example of functions f and g for which the inequality is strict.

2. Decide whether each set is (a) countable, (b) open, (c) closed, (d) bounded, (e) connected, (d) compact. Briefly justify each statement.

   i. $\{ (x,y) : \frac{x^2}{81} + y^2 \leq 1 \text{ and } x^2 + \frac{y^2}{81} < 1\}$

   ii. $\{ x \quad [0,1] : x$ has no 7's in its decimal expansion$\}$

   iii. $\{ (x,y) :$ both x and y are rational $\}$

   iv. $\{(x,y) :$ at least one of x, y is irrational $\}$

3. Determine whether each of the following sequences converges or diverges. Calculate the limit of each convergent sequence.

   a. $x_n = (1 + \frac{1}{4n})^{n-1}$

   b. $y_1 = 197, \quad y_{n+1} = 8 - \frac{7}{y_n}$

   c. $z_n = (3 + \frac{1}{n})^4$

4. Let (x_n) be a sequence of real numbers, and define the *Cesaro means* as the averages $s_N = \frac{1}{N} \sum_{i=1}^{N} x_n$.

a. Show that if (x_n) converges to x, then the sequence of means converges to x as well.

b. Give an example of a bounded, divergent sequence of real numbers (x_n) whose sequence of means converges.

5. Give an ε - δ proof to show that $f(x) = x^3 - 5x + 2$ is continuous at every real number a.

6. a. Suppose f is uniformly continuous on the domain D and that (x_n) is a Cauchy sequence in D. Prove that $(f(x_n))$ is a Cauchy sequence. Please notice that we are not assuming that D is closed, so you may not assume that (x_n) converges to a point at which f is defined and continuous.

b. Give an example of a function f which is bounded and continuous on the open interval (0,1) and a Cauchy sequence (x_n) in (0,1) such that $(f(x_n))$ is not a Cauchy sequence.

7. Let $f_n(x) = n x e^{-nx}$. On the interval $[0,\infty)$, the sequence (f_n) converges pointwise to a function f. What is this function? Is the convergence uniform on $[0,\infty)$? Is it uniform on the interval $[2,\infty)$?